1. $\Delta u = \sin\varphi,\ r < 3,$
   $u(3,\varphi) = \cos 2\varphi.$

2. Найти решение уравнения Лапласа в прямоугольнике $0 < x < \pi,\ 0 < y < \dfrac{\pi}{2}$, если
   $u(0,y) = 0, u_x(\pi,y) = 0, u(x,0) = 0, u(x,\dfrac{\pi}{2}) = \sin\dfrac{3}{2}x$

3. Решить $\Delta u = 0, -\infty < x < \infty, 0, < y < 2\pi; u(x,0) = 0,$
   $u(x,2\pi) = 1,$ при $x < 1; u(x,2\pi) = 0,$ при $x \geq 1$

4. $$\begin{cases} u_{tt} = u_{xx} + e^t \cos 9x + t,\ t > 0,\ 0 < x < \pi; \\ u_x(0,t) = 0, \quad u_x(\pi,t) = 2\pi, \\ u(x,0) = 4\cos 5x + x^2, \\ u_t(x,0) = -1. \end{cases}$$

5. $u_{tt} = \dfrac{1}{4}u_{xx},\ 0 < x < \infty, t > 0$

$u_x(0,t) = 0$, $u(x,0) = \begin{cases} \sin x, x \in [0;\pi] \\ 0, x \notin [0;\pi] \end{cases}$, $u_t(x,0) = 0$. Найти $u(x, 7\pi/4)$ и нарисовать график.

6. $u_{tt} = u_{xx}, t > 0, x > 0;$
   $u(0,t) = 0 \quad u(x,0) = 2\sin x \quad u_t(x,0) = 2\cos x.$

Описать процесс колебаний. Построить профиль струны в момент времени $t = \dfrac{\pi}{4}$.

$$\begin{cases} \Delta u = \sin\varphi & r<3. \\ u(3,\varphi) = \cos 2\varphi. \end{cases}$$

Ищем в виде:

$R(r)\sin\varphi$:

$$\cancel{\sin\varphi}\,\frac{1}{r}(rR')' - \frac{1}{r^2}\cancel{\sin\varphi} = \cancel{\sin\varphi}$$

$$r(R' + rR'') - R = r^2.$$

$$r^2R'' + rR' - R = r^2$$

$$R = y(t)$$

$$rR' = \dot y,\quad r = e^t$$

$$r^2R'' = \ddot y - \dot y$$

$$\ddot y - \dot y + \dot y - y = e^{2t}$$

$$\ddot y - y = e^{2t}$$

$$y = C_1e^t + C_2e^{-t}$$

$$y = ae^{2t}$$

$$4ae^{2t} - ae^{2t} = e^{2t}$$

$$a = 1/3.$$

$$y = C_1e^t + C_2e^{-t} + 1/3\,e^{2t} = 1/3\,e^{2t} = 1/3\,r^2.$$

$$\boxed{u_2 = \frac{1}{3}r^2\sin\varphi.}$$

$$u = u_1 + 1/3\,r^2\sin\varphi.$$

$$\Delta u_1 = 0,\quad r<3.$$

$$u(3,\varphi) = u_1(3,\varphi) + 1/3\cdot 9\sin\varphi = \cos 2\varphi.$$

$$\boxed{u_1(3,\varphi) = \cos 2\varphi - 3\sin\varphi.}$$

Реш. для круга:

$$u(x,t) = A + \cancel{B\ln 2} + \sum_{n=1}^{\infty} 2^n (A_n \cos n\varphi + B_n \sin n\varphi) + \cancel{\sum_{n=1}^{\infty} 2^{-n}(a_n \cos n\varphi + b_n \sin n\varphi)}.$$

Т.к. круг

$$u(x,t) = A + \sum_{n=1}^{\infty} 2^n (A_n \cos n\varphi + B_n \sin n\varphi).$$

$u(3,\varphi) = \cos 2\varphi - 3\sin\varphi$:

$$A + \sum_{n=1}^{\infty} 3^n (A_n \cos n\varphi + B_n \sin n\varphi) = \cos 2\varphi - 3\sin\varphi.$$

Отсюда:

$A = 0$. $A_2 = \frac{1}{3^2} = 1/9$, $B_2 = 0$.

$\cancel{B_3 = -\frac{3}{3^3} = -\frac{1}{9}, A_3 = 0}$.

~~Тогда~~ $\cancel{u(x,t) = 2^2 \cdot 1/9 \cos 2\varphi + 2^3 \cdot (-1/9) \sin}$

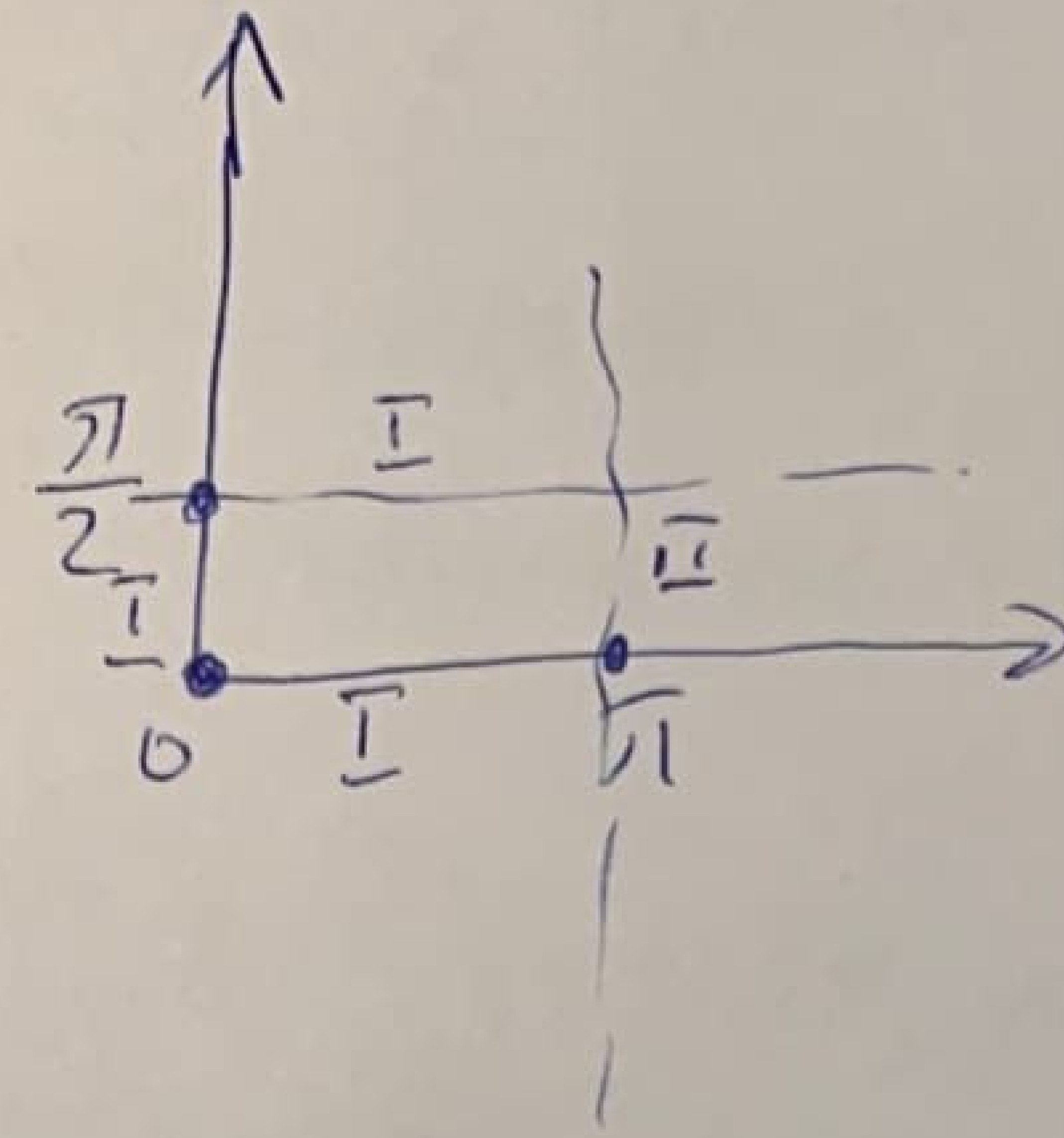
$3B_1 = -3$ $B_1 = -1$.

$u(x,t) = 2^2 \cdot 1/9 \cos 2\varphi - 2 \cdot \sin\varphi$.

+ прибавляем что раньше нашли.

$$\boxed{u(x,t) = 2^2 \cdot 1/9 \cos 2\varphi - 2\sin\varphi + 1/3\, 2^2 \sin\varphi.}$$

$$\begin{cases} \Delta u = 0 \\ u(0,y) = 0 \\ u_x(\pi,y) = 0 \\ u(x,0) = 0 \\ u(x,\pi/2) = \sin 3/2\, x \end{cases}$$



①

$$\begin{cases} \Delta u = 0 \\ u(0,y) = 0 \\ u_x(\pi,y) = 0 \\ u(x,0) = 0 \\ u(x,\pi/2) = \sin 3/2\, x \end{cases}$$

$$X''Y + Y''X = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$

$X'' = -\lambda X$ Задача. I–II: $\lambda_n = \left(\frac{\cancel{\pi}(1+2n)}{2}\right)^2$

$$X_n = \sin\left(\frac{1+2n}{2}\right)x$$

$$Y'' - \lambda Y = 0$$

$$Y = e^{ky}$$

$$k = \frac{1+2n}{2}$$

$$Y_n = A_n e^{\frac{1+2n}{2}y} + B_n e^{-\frac{1+2n}{2}y}$$

$$u(x,y) = \sum_{n=1}^{\infty}\left(A_n e^{\frac{1+2n}{2}y} + B_n e^{-\frac{1+2n}{2}y}\right)\cdot \sin\left(\frac{1+2n}{2}\right)x$$

② $\Delta u = 0$. $u(x,0) = 0$

$u(x,\pi/2) = \sin 3/2\, x$

Отсюда:

$$\begin{cases} A_n + B_n = 0 \\ \left(A_n e^{\frac{1+2n}{2}\pi/2} + B_n e^{-\frac{1+2n}{2}\frac{\pi}{2}}\right)\sin\left(\frac{1+2n}{2}\right)x = \sin 3/2\, x \end{cases}$$

для $n=1$ ⇔ 1, для ост. n $A_n = B_n = 0$

т.к. круг

$$A_1 e^{\frac{3\pi}{4}} + B_1 e^{-\frac{3\pi}{4}} = 1.$$

$A_1 + B_1 = 0.$ $A_1 = -B_1.$

$$-B_1 e^{\frac{3\pi}{4}} + B_1 e^{-\frac{3\pi}{4}} = 1.$$

$$B_1\left(e^{-3\pi/4} - e^{3\pi/4}\right) = 1 \Rightarrow B = \frac{1}{e^{-3\pi/4} - e^{3\pi/4}}.$$

$$A = -B = \frac{1}{e^{3\pi/4} - e^{-3\pi/4}}.$$

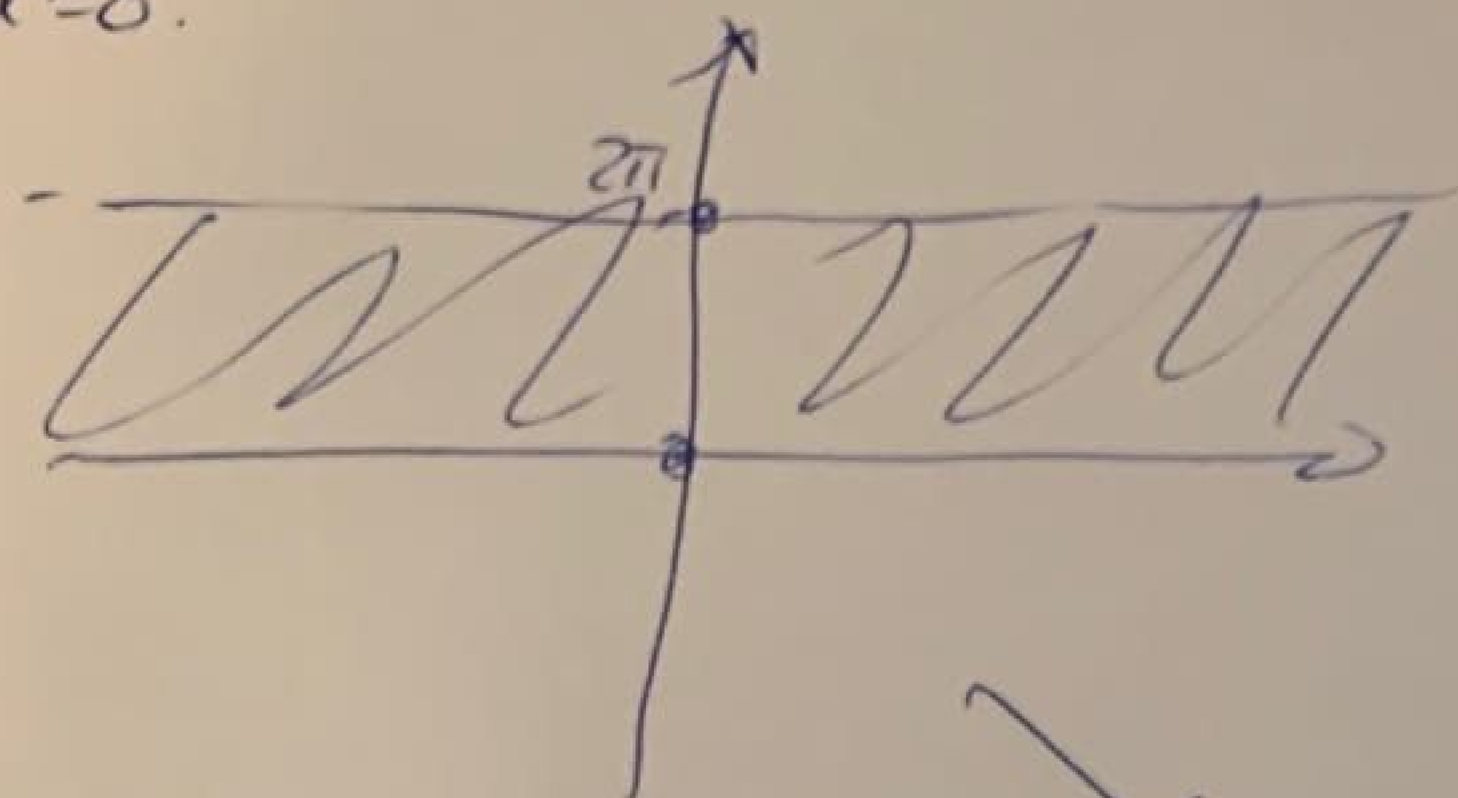
Тогда $u(x,y) = \left(\frac{1}{e^{-3\pi/4} - e^{3\pi/4}} \cdot e^{-3/2y} + \frac{1}{e^{3\pi/4} - e^{-3\pi/4}} e^{3/2y}\right) \cdot \sin\frac{3}{2}x$ (!)

(2) $\begin{cases} \Delta u = 0. \\ u(0,y) = 0 \\ u_x(\pi,y) = 0. \end{cases}$ $u \equiv 0$ подх.

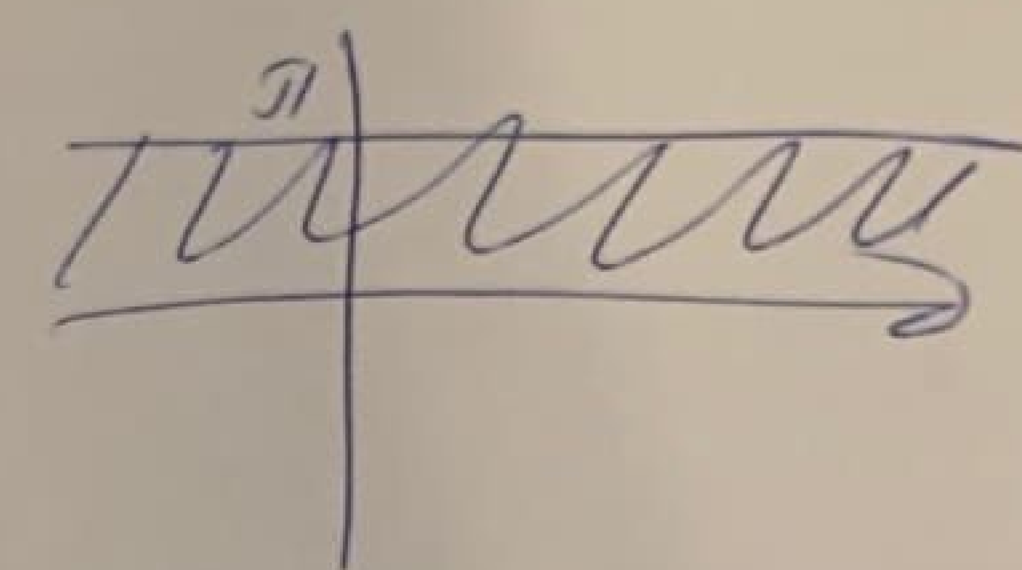
Поэтому общее реш $\equiv$

Ответ:

③. $\Delta u = 0$.



Отобразим сначала



$W_1(z_0) = \frac{1}{2} z_0$

$W_2(w_0) = \cancel{\frac{1}{2} e^{x_0}}\; e^{\frac{1}{2} x_0} \cdot e^{1/2 i y_0}$

$$\begin{cases} \xi_0 = e^{1/2 x_0} \cos y_0 \\ \eta_0 = e^{1/2 x_0} \sin y_0. \end{cases}$$

$e^{\pi/2}$

Решение для полуплоскости:

$V(w_0) = \frac{\eta_0}{\pi} \int_{-\infty}^{\infty} \frac{F(\xi)}{(\xi-\xi_0)^2 + \eta_0^2} d\xi$, где $\cancel{F(\xi)\in}$ $V|_{\eta=0} = \{0, \xi \ge 0$

$e^{1/2 x} = \xi + i\eta_0$.

$w(z) = -e^{1/2 x} \in \cancel{(-\infty, -e^{1/2})}$

$\in (-e^{1/2}, 0)$

$$\begin{cases} F(\xi) = 1, & \xi \in (-e^{1/2}, 0) \\ 0, & \text{иначе} \end{cases}$$

Тогда $V = \frac{\eta_0}{\pi} \left(\frac{1}{\eta_0} \operatorname{arctg} \frac{\xi - \xi_0}{\eta_0} \Big|_0^{-e^{1/2}} \right) =$

Ответ: $= \frac{1}{\pi} \left(\operatorname{arctg} \left(\frac{-e^{1/2} - e^{1/2 x_0} \cos y_0}{e^{1/2 x_0} \sin y_0} \right) - \operatorname{arctg} \left(\frac{-e^{1/2 x_0} \cos y_0}{e^{1/2 x_0} \sin y_0} \right) \right)$

6.3 Рисем. вепемог згодому:

$$\begin{cases} u_{tt} = u_{xx}, \ t>0, \ x\in R \\ u(x,0) = 2\sin x. \\ u_t(x,0) = \begin{cases} 2\cos x, & x>0 \\ -2\cos x, & x\le 0 \end{cases} = \varphi(x) \end{cases}$$

$$u(x,t) = \sin(x-t)+\sin(x+t) + \frac{1}{2}\int_{x-t}^{x+t}\varphi(z)\,dz$$

$$\frac{1}{2}\left(\int_{x-t}^{0} -2\cos x + \int_{0}^{x+t} 2\cos x\,dx\right) = -\sin x\Big|_{x-t}^{0} + \sin x\Big|_{0}^{x+t} = \sin(x+t)+\sin(x-t)$$

$$\frac{1}{2}\int_{x-t}^{x+t} 2\cos x\,dx = \sin x\Big|_{x-t}^{x+t} = \sin(x+t)-\sin(x-t)$$

$$u(x,t) = \begin{cases} 2(\sin(x-t)+\sin(x+t)) = 4\sin x\cos t, & 0<x<t \\ 2\sin(x+t), & x\ge t \end{cases}$$

$$u\left(x,\frac{\pi}{4}\right) = \begin{cases} 4\sin x\cdot\frac{\sqrt{2}}{2} = 2\sqrt{2}\sin x, & 0<x<\frac{\pi}{4} \\ 2\sin\left(x+\frac{\pi}{4}\right), & x\ge\frac{\pi}{4} \end{cases}$$